JOHANNES KEPLER UNIVERSITÄT LINZ
INSTITUT FÜR ANGEWANDTE GEOMETRIE
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7. Februar 2014

1. Übung

zum 3. Schülerseminar
über komplexe Zahlen

Grundlagen

$a = -6 + 8\mathrm{i}$, $b = 4 + 3\mathrm{i}$

Aufgabe 1:

Wende die Grundrechnungsarten zum Berechnen der komplexen Zahlen an:
a) $z_0 = a + b$ b) $z_1 = a - b$ c) $z_2 = a \cdot b$ d) $z_3 = \frac{a}{b}$

Aufgabe 2:

Berechne den Betrag und die komplex Konjugierte:
a) $z_4 = |b|$ b) $z_5 = \overline{a}$

Aufgabe 3:

Löse die quadratische Gleichung aus dem Theorieteil und mache die Probe durch Einsetzen.
a) $x^2 - 2x + 5 = 0$.

***Aufgabe 4:**

a) $(3 + 4\mathrm{i}) \cdot \overline{(3 + 4\mathrm{i})} = |(3 + 4\mathrm{i})|^2$
b) Zeige, dass immer gilt: $z \cdot \overline{z} = |z|^2$

***Aufgabe 5:**

Löse folgende Gleichung:
$(3 + 4\mathrm{i})^2 - 2(x - \mathrm{i}y) = x + \mathrm{i}y$

***Aufgabe 6:**

Finde die Wurzeln von $35 - 12\mathrm{i}$.

Zusammenfassung:

- $z = \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z)$
- $(a + b\mathrm{i}) = (c + d\mathrm{i}) \Leftrightarrow a = c \wedge b = d$
- $(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$
- $(a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}$
- $(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$
- $\dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} \rightarrow$ erweitern mit der konjugiert Komplexen.
- $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$
- $\overline{a + b\mathrm{i}} = a - b\mathrm{i}$

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2. Übung

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Geometrie I

Aufgabe 1:

Let $z = x + y\mathrm{i} = r\cos\varphi + r\sin\varphi\mathrm{i}$.

a) Express r and φ in terms of x and y. (hint: express $\frac{y}{x}$)
b) Write the following complex numbers in mod-arg form.

$$z_1 = 1, \quad z_2 = 2\mathrm{i}, \quad z_3 = 2 + 2\mathrm{i}, \quad z_4 = -1 + \sqrt{3}\mathrm{i}$$

Aufgabe 2:

Let A, B, C and D be four complex numbers that form a positively oriented square (vertices are marked anticlockwise).
a) Express complex numbers C and D in terms of A and B.
b) Find vertices C and D if $A = -2\mathrm{i}$, $B = 4 + \mathrm{i}$ and compute the length of the square's site.

Aufgabe 3:

<u>Theorem</u>
Let A, B and C be three complex numbers that form a positively oriented triangle $\triangle ABC$. Let $\triangle ABD$ be an equilateral triangle above the site AB and $\triangle BCE$ be an equilateral triangle above site BC. Mark the following points: F is the midpoint of BD, G is the midpoint of BE, H is the midpoint of AC.

Then $\triangle FGH$ is an equilateral triangle.

Verify the Theorem by drawing it in GeoGebra and find the length of the site in the triangle $\triangle FGH$ if $A = -2$, $B = 4 + \mathrm{i}$ and $C = 4\mathrm{i}$.

Aufgabe 4:

<u>Abel's Theorem</u>:

Let $\square ABCD$ be a general quadrilateral (ein beliebiges Viereck). Draw a square (Quadrat) above each site and mark the centers of the four squares by K, L, M and N. Connect the centers of the opposite squares by lines.

The connecting lines are perpendicular and of the same length.

Prove the Abel's Theorem and verify it using GeoGebra. (hint: use the theorem from the slides)

Summary

mod-arg form: $z = r\cos\varphi + r\sin\varphi \mathrm{i}$

r	distance from the origin (length of the vector z)
φ	angle between real axis and the vector z
$\lvert z\rvert$	distance from the origin (length of the vector z)
$\bar{z}$	reflection of vector z across x-axis
$z + a$	vector obtained by adding vectors z and a
$z - a$	vector obtained by subtracting vectors z and a
$z \cdot \mathrm{i}$	rotation of vector z around the origin by 90°
$z \cdot (-1)$	rotation of vector z around the origin by 180°
$z \cdot (-\mathrm{i})$	rotation of vector z around the origin by 270°

Theorem:
Let A, B, C be complex numbers that form a positively oriented triangle. Let D be a complex number, such that $D = C + (B - A) \cdot \mathrm{i}$. If p is a line connecting A and B and q is a line connecting C and D, then p and q are perpendicular (senkrecht aufeinander stehen) and of the same length.

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3. Übung

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Geometrie II

Aufgabe 1:

Die komplexen Zahlen a, b, c und d haben ein reelles Doppelverhältnis, daher $DV(a,b,c,d) = \lambda \in \mathbb{R}$.

Berechne:
a) $DV(b,a,c,d)$
b) $DV(c,d,a,b)$

Aufgabe 2:

Gegeben sind fünf komplexe Zahlen a, a', b, c und d.

Zeige, dass gilt:
Falls a, b, c und d auf einem Kreis liegen und falls auch a', b, c und d auf einem Kreis liegen, dann liegen auch a, a', c und d auf einem Kreis.

Aufgabe 3:

Satz von Miquel

Gegeben sind acht komplexe Zahlen P_0, P_1, P_2, P_3, P_4, P_5, P_6 und P_7.
P_0, P_1, P_2 und P_3 liegen auf dem Kreis c_0.
P_0, P_1, P_4 und P_7 liegen auf dem Kreis c_1.
P_1, P_2, P_4 und P_5 liegen auf dem Kreis c_2.
P_2, P_3, P_5 und P_6 liegen auf dem Kreis c_3.
P_4, P_5, P_6 und P_7 liegen auf dem Kreis c_4.

a) Zeige, dass dann auch P_0, P_3, P_6 und P_7 auf einem Kreis c_5 liegen.
b) Verifiziere den Satz von Miquel durch eine Konstruktion in GeoGebra.

Zusammenfassung:

- $\mathsf{DV}(a,b,c,d) = \dfrac{a-c}{a-d} : \dfrac{b-c}{b-d}$
- a, b, c, d liegen auf einem Kreis* $\Leftrightarrow$ $\mathsf{DV}(a,b,c,d)$ ist reell.
- Kreisgleichung: $p, q \in \mathbb{R}$, $p \cdot q \leq 1$, $u \in \mathbb{C}$, $u \cdot \bar{u} = 1$

$$p \cdot (z \cdot \bar{z}) - \bar{u} \cdot z - u \cdot \bar{z} + q = 0$$

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Lösungen

zum 3. Schülerseminar
über komplexe Zahlen

1. Übung

$a = -6 + 8\mathrm{i}, \quad b = 4 + 3\mathrm{i}$

Aufgabe 1:

a) $z_0 = -2 + 11\mathrm{i}$ b) $z_1 = -10 + 5\mathrm{i}$ c) $z_2 = -48 + 14\mathrm{i}$ d) $z_3 = 2\mathrm{i}$

Aufgabe 2:

a) $z_4 = 5$ b) $z_5 = -6 - 8\mathrm{i}$

Aufgabe 3:

$x_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4\mathrm{i}}{2} = 1 \pm 2\mathrm{i} \quad \Rightarrow \quad x_1 = 1 + 2\mathrm{i}, \; x_2 = 1 - 2\mathrm{i}$
$x_1 : (1 + 2\mathrm{i})^2 - 2(1 + 2\mathrm{i}) + 5 = 1 + 4\mathrm{i} - 4 - 2 - 4\mathrm{i} + 5 = 0$
$x_2 : (1 - 2\mathrm{i})^2 - 2(1 - 2\mathrm{i}) + 5 = 1 - 4\mathrm{i} - 4 - 2 + 4\mathrm{i} + 5 = 0$

Aufgabe 4:

a) $(3 + 4\mathrm{i})\overline{(3 + 4\mathrm{i})} = (3 + 4\mathrm{i})(3 - 4\mathrm{i}) = 3^2 - (4\mathrm{i})^2 = 9 + 16 = 25$
$|3 + 4\mathrm{i}|^2 = 3^2 + 4^2 = 9 + 16 = 25$
b) $z = x + y\mathrm{i}$
$z \cdot \bar{z} = (x + y\mathrm{i})(x - y\mathrm{i}) = x^2 - (y\mathrm{i})^2 = x^2 + y^2$
$|z|^2 = x^2 + y^2$

Aufgabe 5:

$(3 + 4\mathrm{i})^2 - 2(x - y\mathrm{i}) = x + y\mathrm{i}$
$-7 - 2x + (24 + 2y)\mathrm{i} = x + y\mathrm{i}$
By comparing the real and imaginary parts on both sides, we get:
$-7 - 2x = x \quad \Rightarrow \quad x = -\frac{7}{3}$
$24 + 2y = y \quad \Rightarrow \quad y = -24$

Aufgabe 6:

$\sqrt{35 - 12\mathrm{i}} = z = x + y\mathrm{i}, \quad x, y \in \mathbb{R}$
$35 - 12\mathrm{i} = (x + y\mathrm{i})^2 = x^2 - y^2 + 2xy\mathrm{i}$
By comparing the real and imaginary parts on both sides we get two equations:
$x^2 - y^2 = 35$
$2xy = -12$
Express y with respect to x in the second equation and plug it in the first:
$y = -\frac{6}{x} \quad \Rightarrow \quad x^2 - \frac{36}{x^2} - 35 = 0$
We multiply by x^2 and get a quadratic equation for x^2:
$x^4 - 35x^2 - 36 = 0$
$(x^2 - 36)(x^2 + 1) = 0$
$(x^2)_1 = 36 \quad \Rightarrow \quad x_{1,2} = \pm 6 \quad \Rightarrow \quad y_{1,2} = \mp 1$
$(x^2)_2 = -1 \quad \Rightarrow \quad x_{3,4} = \pm \mathrm{i}$ \\ $x_{3,4}$ are not the right solutions, since $x \in \mathbb{R}$.
Therefore the only solutions are: $z_1 = 6 - \mathrm{i}, \quad z_2 = -6 + \mathrm{i}$.

2. Übung

Aufgabe 1:

$z = x + y\mathrm{i} = r\cos\varphi + r\sin\varphi\mathrm{i}$

a) $r = |z| = \sqrt{x^2 + y^2}$

$\frac{y}{x} = \frac{r\cos\varphi}{r\sin\varphi} = \tan\varphi \quad \Rightarrow \quad \varphi = \arctan\frac{y}{x}$

b) The solutions are:

$z_1 : r = 1,\ \varphi = 0 \quad \Rightarrow \quad z_1 = \cos 0 + \sin 0\mathrm{i}$

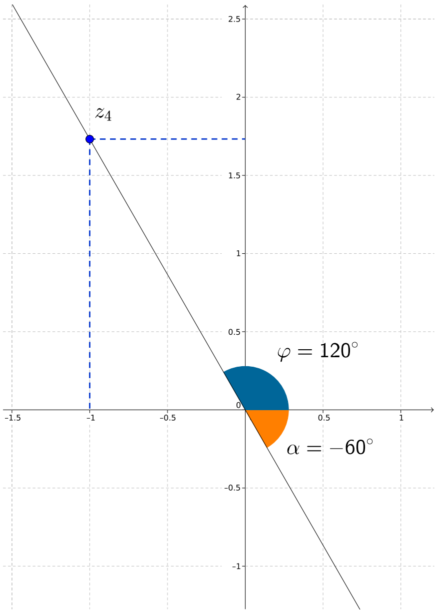
$z_2 : r = 2,\ \varphi = 90° \quad \Rightarrow \quad z_2 = 2\cos 90° + 2\sin 90°\mathrm{i}$

$z_3 : r = \sqrt{8},\ \tan\varphi = 1 \quad \Rightarrow \quad \varphi = 45° \quad \Rightarrow \quad z_3 = \sqrt{8}\cos 45° + \sqrt{8}\sin 45°\mathrm{i}$

$z_4 : r = 2,\ \tan\varphi = -\sqrt{3}$

The angle φ we want to compute is the one marked in blue on the picture. The angle that corresponds to the computed tangens $-\sqrt{3}$ is $-60°$. But if you look at the picture, this angle is marked in orange, and this is not the right angle we are looking for. The right φ is computed by $180° - 60°$ which is $120°$:

$z_4 = 2\cos 120° + 2\sin 120°\mathrm{i}$



Aufgabe 2:

a) $C = B + (B - A)\mathrm{i},\ D = A + (B - A)\mathrm{i}$

b) $C = 1 + 5\mathrm{i},\ D = -3 + 2\mathrm{i}$, length $= 5cm$

Aufgabe 3:

length $= 5.5cm$

Aufgabe 4:

We can prove the theorem by proving the relation between the four centers as written in the theorem from the slides: If we prove that the fourth center is expressed in terms of other three as in the theorem from the slides (note the orientation!), we prove that the lines connecting the centers of the opposite squares are perpendicular and of the same length. For example: If you mark the centers in the anticlockwise manner as K, L, M, N with K being the center of the square above A and B, then we shall verify if the following equation holds:

$$L = N + (K - M)\mathrm{i}. \tag{1}$$

We express the centers in terms of the four corners A, B, C, D as
$K = B + \frac{A-B}{2} + \frac{A-B}{2}\mathrm{i}$,
$L = C + \frac{B-C}{2} + \frac{B-C}{2}\mathrm{i}$,
$M = D + \frac{C-D}{2} + \frac{C-D}{2}\mathrm{i}$,
$N = A + \frac{D-A}{2} + \frac{D-A}{2}\mathrm{i}$
and plug them in the equation (1). We verify that the equation (1) is correct.

We can prove the Abel's Theorem without using the theorem from the slides: We want to verify if a complex number $(L - N)$ can be expressed as a rotation of the vector $(K - M)$ by $90°$, which means we want to see if the following equation is correct:

$$(L - N) = (K - M)\mathrm{i}. \tag{2}$$

We now plug the expressions for K, L, M and N (written above) into equation (2). The equation holds and this means that one line between the centers can be expressed as the other line rotated by $90°$, so they are perpendicular and of the same length.

3. **Übung**

Aufgabe 1:

a) $DV(b,a,c,d) = \frac{\frac{b-c}{b-d}}{\frac{a-c}{a-d}} = \left(\frac{\frac{a-c}{a-d}}{\frac{b-c}{b-d}}\right)^{-1} = \lambda^{-1} = \frac{1}{\lambda}$

b) $DV(c,d,a,b) = \frac{\frac{c-a}{c-b}}{\frac{d-a}{d-b}} = \frac{\frac{a-c}{b-c}}{\frac{a-d}{b-d}} = \frac{\frac{(a-c)(a-d)}{(b-c)(a-d)}}{\frac{(a-d)(b-c)}{(b-d)(b-c)}} = \lambda \cdot \frac{\frac{a-d}{b-c}}{\frac{a-d}{b-c}} = \lambda$

Aufgabe 2:

We know that:

$DV(a,b,c,d) = \frac{\frac{a-c}{a-d}}{\frac{b-c}{b-d}} = \lambda \in \mathbb{R}$

$DV(a',b,c,d) = \frac{\frac{a'-c}{a'-d}}{\frac{b-c}{b-d}} = \mu \in \mathbb{R}$

Write $DV(a,a',c,d)$ and plug in the known fractions, expressed in terms of λ and μ:

$DV(a,a',c,d) = \frac{\frac{a-c}{a-d}}{\frac{a'-c}{a'-d}} = \frac{\lambda\,\frac{b-c}{b-d}}{\mu\,\frac{b-c}{b-d}} = \frac{\lambda}{\mu} \in \mathbb{R}$

Aufgabe 3:

We have to prove that P_0, P_3, P_6, P_7 lie on a circle, that is, their cross ratio (DV) is real. We will prove that

$$c_5 : DV(P_3,P_7,P_0,P_6) = \frac{P_3-P_0}{P_3-P_6} : \frac{P_7-P_0}{P_7-P_6} = \lambda_5 \in \mathbb{R} \qquad (3)$$

We can express all the cross ratios of other points on the circles in the following way:

$c_0 : DV(P_3,P_1,P_0,P_2) = \frac{P_3-P_0}{P_3-P_2} : \frac{P_1-P_0}{P_1-P_2} = \lambda_0 \in \mathbb{R}.$

$c_1 : DV(P_7,P_1,P_0,P_4) = \frac{P_7-P_0}{P_7-P_4} : \frac{P_1-P_0}{P_1-P_4} = \lambda_1 \in \mathbb{R}.$

$c_2 : DV(P_1,P_5,P_4,P_2) = \frac{P_1-P_4}{P_1-P_2} : \frac{P_5-P_4}{P_5-P_2} = \lambda_2 \in \mathbb{R}.$

$c_3 : DV(P_3,P_5,P_6,P_2) = \frac{P_3-P_6}{P_3-P_2} : \frac{P_5-P_6}{P_5-P_2} = \lambda_3 \in \mathbb{R}.$

$c_4 : DV(P_7,P_5,P_6,P_4) = \frac{P_7-P_6}{P_7-P_4} : \frac{P_5-P_6}{P_5-P_4} = \lambda_4 \in \mathbb{R}.$

We express nominators and denominators in equation (3) from the five equations above:

$P_3 - P_0 = \frac{\lambda_0(P_3-P_2)(P_1-P_0)}{(P_1-P_2)}$

$$P_3 - P_6 = \frac{\lambda_3(P_3 - P_2)(P_5 - P_6)}{(P_5 - P_2)}$$
$$P_7 - P_0 = \frac{\lambda_1(P_7 - P_4)(P_1 - P_0)}{(P_1 - P_4)}$$
$$P_7 - P_6 = \frac{\lambda_4(P_7 - P_4)(P_5 - P_6)}{(P_5 - P_4)}$$

After crossing out the same expressions in the nominator and denominator, we get the following:

$$\lambda_5 = \frac{\lambda_0\lambda_4(P_5 - P_2)(P_1 - P_4)}{\lambda_3\lambda_1(P_1 - P_2)(P_5 - P_4)} = \frac{\lambda_0\lambda_4\lambda_2}{\lambda_3\lambda_1} \in \mathbb{R}.$$